Moments of Inertia

- Recall center of mass

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

$$M_y = \iint_R x\,\delta(x,y)\,dA$$

distance to y-axis

$$M_x = \iint_R y\,\delta(x,y)\,dA$$

distance to x-axis

- Moments of Inertia I_x, I_y

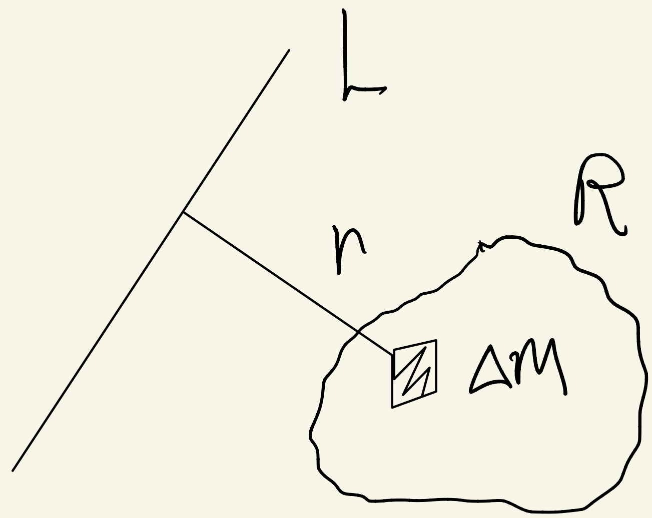
$$I_x = \iint_R y^2\,\delta(x,y)\,dA$$

$$I_y = \iint_R x^2\,\delta(x,y)\,dA$$

"weight the density $\delta(x,y)$ with distance to the axis squared"

Defn: Moment of Inertia about an arbitrary axis L

$$I_L = \iint_R r^2 \, \delta(x,y)\, dA$$



Here $r = r(x,y)$ is a function of (x,y) which must be computed to evaluate the integral.

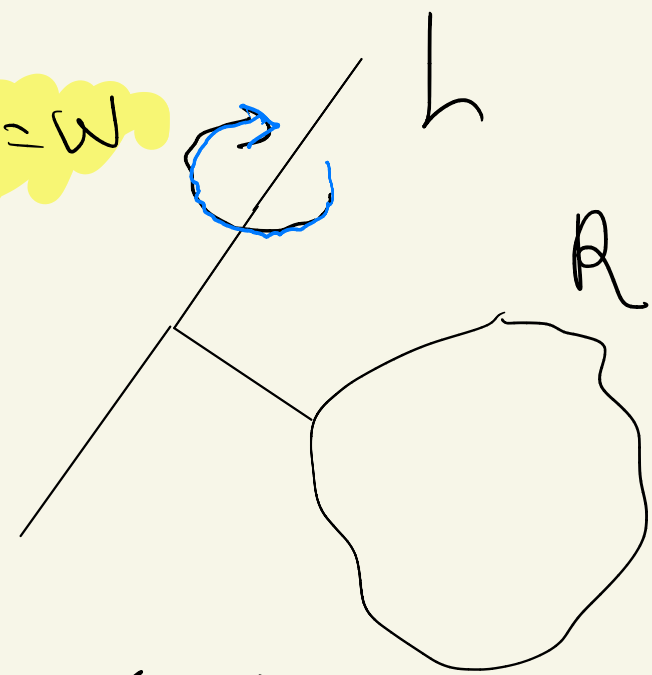
Q: Why are moments of Inertia important?

Ans: (For us) Kinetic Energy of Rotation

Ex: Derive a formula for the Kinetic Energy stored in a metal plate R of density $\sigma(x,y)$ when it is rotated about axis L

Picture:

$$\frac{d\theta}{dt} = \omega$$



Soln: We start with the defn of Kinetic Energy (KE) of a point mass m:

Defn: $KE = \frac{1}{2} m v^2$

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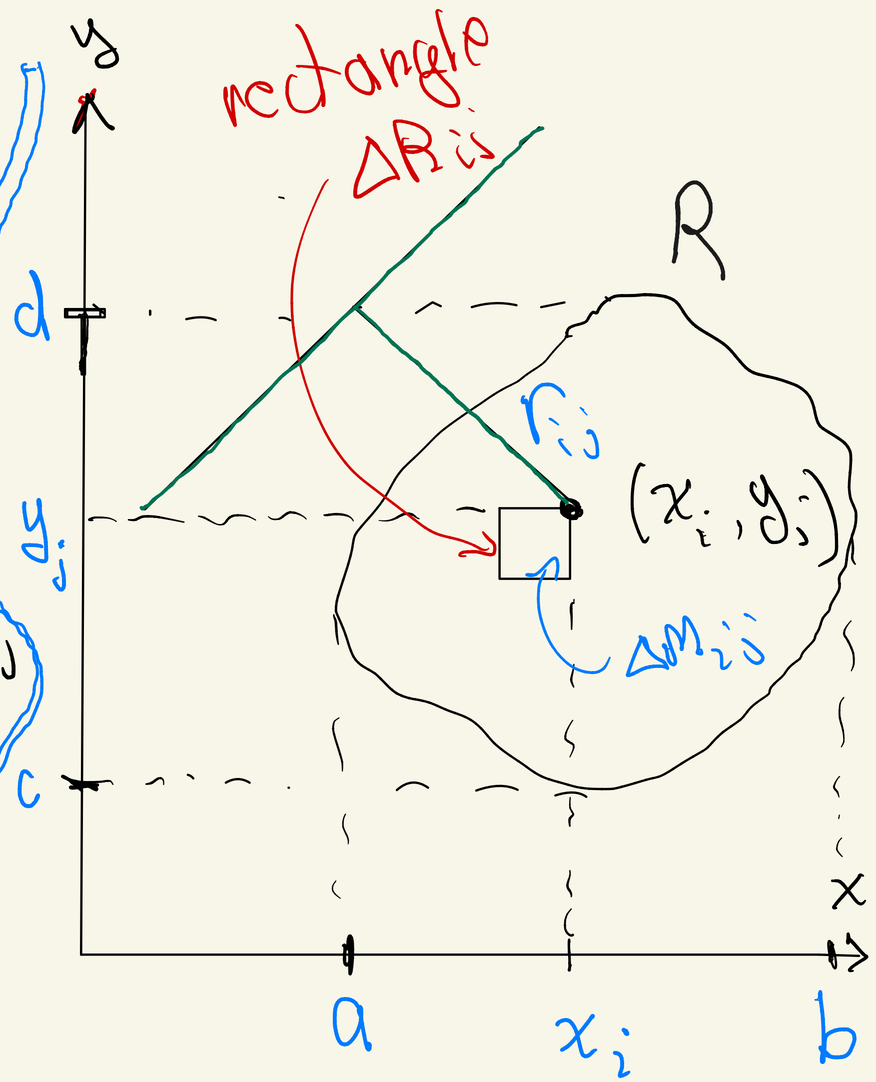
Soln: Idea: approx the KE of the continuously distributed $\delta(x,y)$ by a Riemann Sum & take the limit to get an integral...

- Cover R with grid (x_i, y_j)

- Find KE of mass ΔM_{ij} in ΔR_{ij}



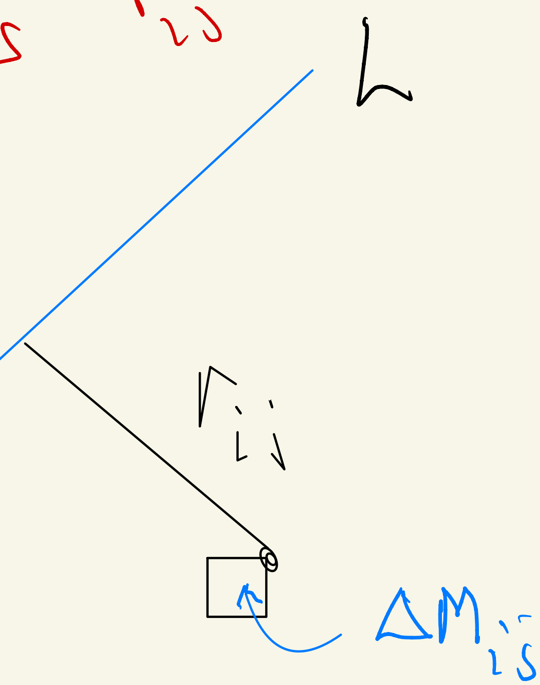
$$KE = \frac{1}{2} M v^2$$

$$M = \Delta M_{ij}$$

$$v = r_{ij} \frac{d\theta}{dt} = r_{ij}\,\omega$$

$$\Delta KE_{ij} = \frac{1}{2}\underbrace{\Delta M_{ij}}_{M}\underbrace{(r_{ij}w)^2}_{V^2}$$

$r_{ij} = r_{ij}(x,y)$



$$\Delta M_{ij} \approx \delta(x_i, y_j)\,\Delta x\,\Delta y$$

$\frac{\text{mass}}{\text{area}} \times \text{area}$

$$\Delta KE_{ij} = \frac{1}{2}\delta(x_i, y_j)\, r_{ij}^2\, w^2\, \Delta x\,\Delta y$$

constant angular rotation rate

constant

$$KE = \lim_{N\to\infty}\sum_{(x_i,y_j)\in R}\left(\frac{1}{2}w^2\right)\delta(x_i,y_j)\, r_{ij}^2\,\Delta x\,\Delta y$$

$$\boxed{KE = \frac{1}{2}w^2\iint_R r^2\,\delta(x,y)\,dx\,dy = \frac{1}{2}I_L w^2}$$

Conclude: The Kinetic Energy of rotation of a continuously distributed mass (metal plate) about axis L is:

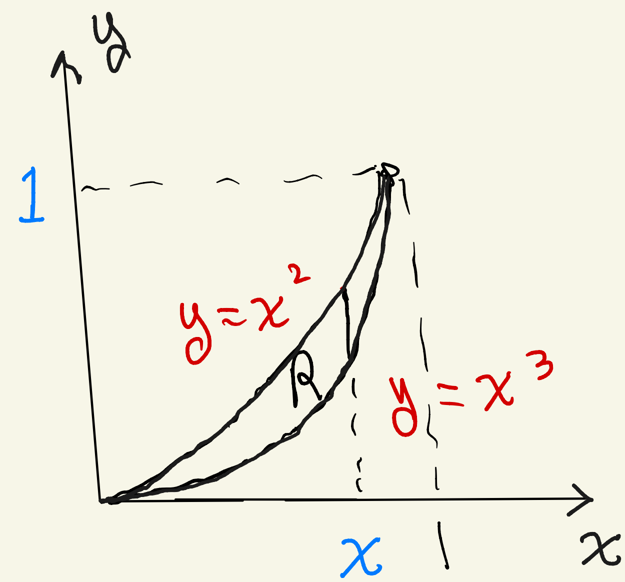
$$KE = \frac{1}{2} I_L \, \omega^2$$

Moment of Inertia about axis L

angular rotation rate

$\omega = \frac{\text{radians}}{\text{sec}}$

Example: Consider a metal plate R of density $\delta(x,y) = 1$ whose shape is the region between $y = x^2$ and $y = x^3$



- Find the center of mass —

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

$$M = \iint_R \delta(x,y)\,dA = \int_0^1 \int_{x^3}^{x^2} 1\,dy\,dx$$

$$= \int_0^1 y\Big]_{x^3}^{x^2} dx = \int_0^1 x^2 - x^3\,dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4}\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\boxed{M = \frac{1}{12}}$$

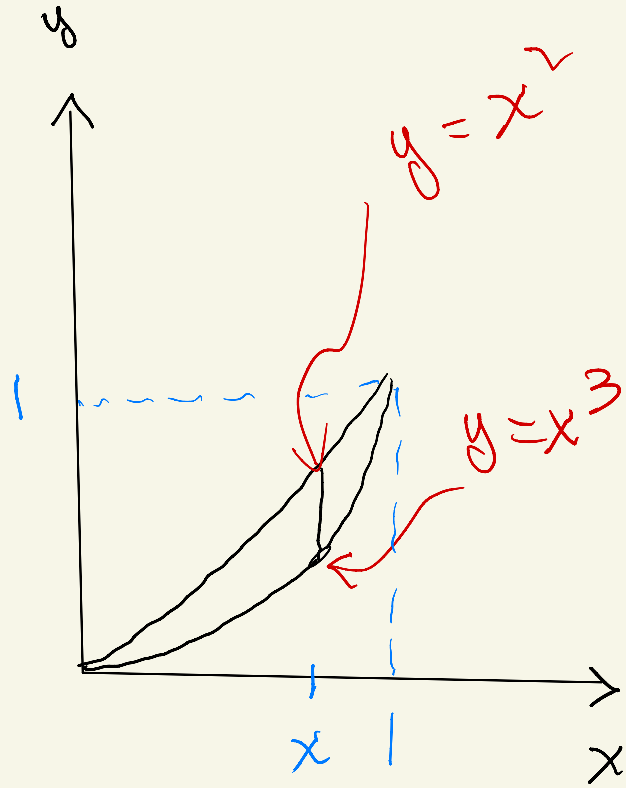
$$M_y = \iint_R x\,\delta(x,y)\,dA$$

$$= \int_0^1 \int_{x^3}^{x^2} x \cdot 1 \, dy\, dx$$

$$= \int_0^1 xy\Big]_{y=x^3}^{x^2} dx$$

$$= \int_0^1 x(x^2 - x^3)\,dx = \int_0^1 x^3 - x^4\,dx$$

$$= \frac{x^4}{4} - \frac{x^5}{5}\Big]_0^1 = \frac{1}{4} - \frac{1}{5} = \boxed{\frac{1}{20}}$$



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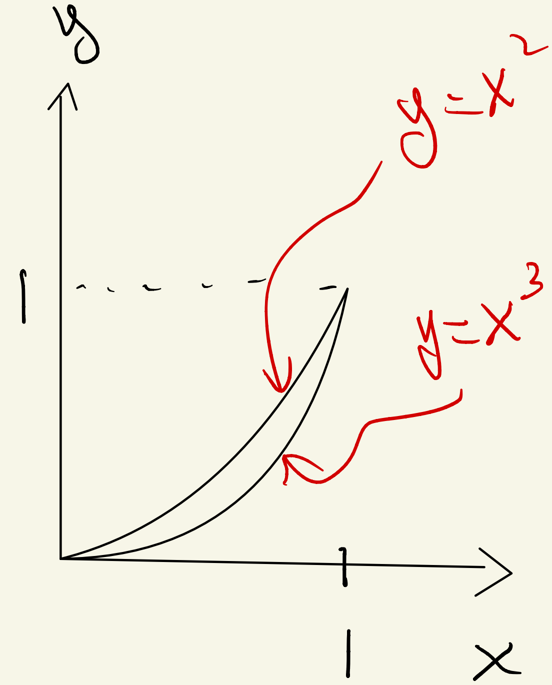
$$\boxed{M = \frac{1}{12} \qquad M_y = \frac{1}{20}}$$

$$M_x = \iint_R y\,\delta(x,y)\,dA$$



$$= \iint_R y \cdot dA$$

$$= \int_0^1 \int_{x^3}^{x^2} y \, dy \, dx = \int_0^1 \frac{y^2}{2} \Big]_{y=x^3}^{x^2} dx$$

$$= \int_0^1 \frac{x^4}{2} - \frac{x^6}{2} \, dx = \frac{x^5}{10} - \frac{x^7}{14} \Big]_0^1$$

$$= \frac{1}{10} - \frac{1}{14} = \frac{7}{70} - \frac{5}{70} = \frac{2}{70} = \boxed{\frac{1}{35}}$$

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Conclude: $M = \frac{1}{12}$, $M_y = \frac{1}{20}$, $M_x = \frac{1}{35}$

Thus —

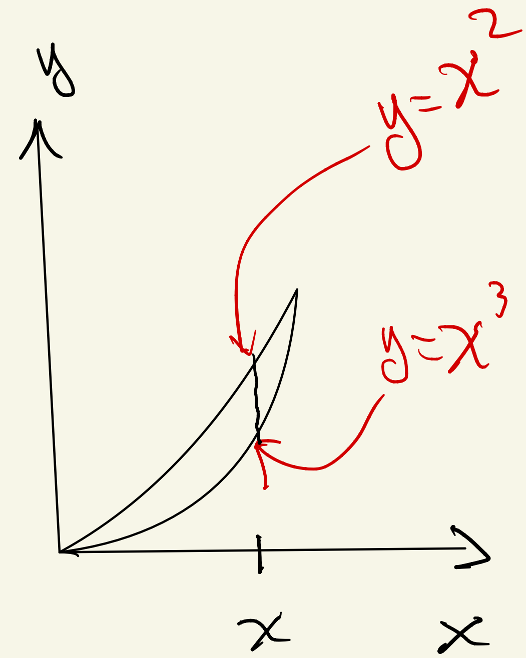
$$\bar{x} = \frac{M_y}{M} = \frac{1/20}{1/12} = \frac{12}{20} = \frac{3}{5}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1/35}{1/12} = \frac{12}{35}$$

Center of mass $(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$

Ex: Same R and $\delta = 1$

Find the KE of rotation of R about the y-axis, assuming a rotation rate of $\omega \; \frac{\text{radians}}{\text{sec}}$



Soln:

$$KE = \frac{1}{2} I_y \omega^2 = \frac{1}{60}\omega^2$$

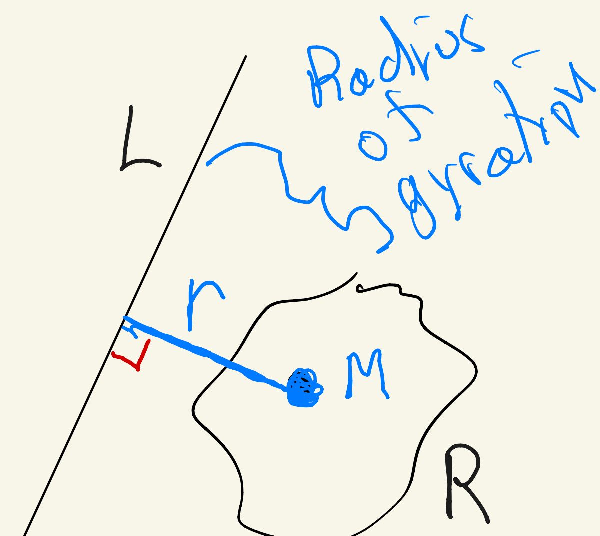
$$I_y = \iint_R x^2 \delta(x,y)\, dA$$

$$= \int_0^1 \int_{x^3}^{x^2} x^2 \, dy \, dx = \int_0^1 x^2 y \Big]_{y=x^3}^{y=x^2} dx$$

$$= \int_0^1 x^2(x^2 - x^3)\, dx = \frac{x^5}{5} - \frac{x^6}{6}\Big]_0^1 = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$$

Ex: Find the radius of gyration of R about the y-axis –

Soln: The radius of gyration is the distance from the axis at which you get the same KE of rotation with all mass at one point –



Let r = radius of gyration. Then

$$\frac{1}{\cancel{2}} M (\underbrace{r\cancel{\omega}}_{\text{speed of } M})^2 = \frac{1}{\cancel{2}} I_L \cancel{\omega}^2$$

$$r^2 = \frac{I_L}{M} \Rightarrow r = \sqrt{\frac{I_L}{M}}$$

For Us : $L \equiv$ y-axis

$I_y = \frac{1}{30}$ $M = \frac{1}{12}$

$$r = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{1/30}{1/12}} = \sqrt{\frac{12}{30}}$$

$$\boxed{r = \sqrt{\frac{6}{15}}}$$

Radius of Gyration